JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 68, 408–420 (1979)

Bifurcation of Periodic Solutions for a Semilinear Wave Equation

HANSJÖRG KIELHÖFER

Institut für Angewandte Mathematik und Statistik der Universität, 8700 Würzburg, West Germany

Submitted by P. D. Lax

Bifurcation of time periodic solutions and their regularity are proved for a semilinear wave equation $u_{tt} - u_{xx} - \lambda u = f(\lambda, x, u)$, $x \in (0, \pi)$, $t \in \mathbb{R}$, together with Dirichlet or Neumann boundary conditions at $x = 0$ and $x = \pi$. The set of values of the real parameter λ where bifurcation from the trivial solution $u = 0$ occurs is dense in $\mathbb{R}$.

INTRODUCTION

In this paper we consider the following one dimensional semilinear autonomous wave equation

$$u_{tt} - u_{xx} - \lambda u = f(\lambda, x, u), \qquad t \in \mathbb{R}, \quad x \in (0, \pi), \quad \lambda \in \mathbb{R}, \tag{0.1}$$

together with either the boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag*{(0.2)$_\mathrm{D}$}$$

or

$$u_x(t, 0) = u_x(t, \pi) = 0. \tag*{(0.2)$_\mathrm{N}$}$$

We are only interested in solutions u of (0.1)–(0.2) which are periodic in t, i.e.

$$u(t + P, x) = u(t, x), \qquad x \in [0, \pi], \quad t \in \mathbb{R}, \quad P > 0. \tag{0.3}$$

Since λ is a free parameter and the period P is not known or prescribed a priori, we call a solution a triple (λ, u, P) satisfying (0.1)–(0.3). We assume $f(\lambda, x, 0) = 0$ so that $u \equiv 0$ is a solution of (0.1)–(0.3) for all λ and P. Following the terminology used in ordinary differential equations we call a *nontrivial* time periodic solution a *free vibration*. In contrast to P. Rabinowitz [4] we are not looking for free vibrations globally but only for those which bifurcate from the trivial solutions $(\lambda, 0, P)$.

Sponsored in part by the United States Army under Contract No. DAAG29-75-C-0024 and in part by the "Deutsche Forschungsgemeinschaft" under No. Ki 227/1.

0022-247X/79/040408–13$02.00/0

Our main result is that under the conditions $f \in C^5(\mathbb{R} \times [0, \pi] \times \mathbb{R}), f(\lambda, x, 0) = f_u(\lambda, x, 0) = 0$, (and also $f_x(\lambda, 0, u) = f_x(\lambda, \pi, u) = 0$ in case of boundary conditions $(0.2)_{\mathrm{N}}$), there is a dense set Λ in $\mathbb{R}$ and a corresponding set of periods such that $(\lambda_0, 0, P_m)$ is a bifurcation point of classical solutions of (0.1)–(0.3) whenever λ_0 is in Λ and $P_m = P_m(\lambda_0)$.

To be more precise, for each $\lambda_0 \in \Lambda$ there locally exists a continuous family $(\lambda(c), u(c), P_m)$ in $\mathbb{R} \times C^2(\mathbb{R} \times [0, \pi]) \times \mathbb{R}_+$ of nontrivial solutions of (0.1)–(0.3) with $(\lambda(0), u(0), P_m) = (\lambda_0, 0, P_m)$, where c denotes a parameter and the period P_m depends only on λ_0 (and not on c), so that the solutions $u = u(c)$ on a fixed branch emanating at $(\lambda_0, 0, P_m)$ all have the same period $P_m = P_m(\lambda_0)$. The reason for the subscript m in P_m is that the period is determined from a linearized problem corresponding to (0.1)–(0.3), and we choose P_m to be minimal among all periods of that problem which are rational multiples of π.

In Section 1 we shall give an implicit definition of Λ as well as an explicit characterization of a dense subset of Λ. We emphasize now that Λ does not contain 0.

It should be observed that any phase shift in t of a solution is again a solution. Thus we get well defined branches only by fixing the phase.

Our main result has the following consequences: Consider instead of (0.1) the equation

$$u_{tt} - u_{xx} - \nu u = \rho g(u), \qquad \nu, \rho \in \mathbb{R}, \tag{0.4}$$

together with the same boundary conditions (0.2) and periodicity (0.3). Here we assume $g(0) = 0$ but $g_u(0) \neq 0$. By transforming (0.4) into

$$u_{tt} - u_{xx} - (\nu + \rho g_u(0))\, u = \rho(g(u) - g_u(0)\, u) \tag{0.5}$$

we get the following results: Fix $\nu = \nu_0 \in \Lambda$. Then there exists a continuous family $(\rho(c), u(c), P_m)$ in $\mathbb{R} \times C^2(\mathbb{R} \times [0, \pi]) \times \mathbb{R}_+$ of nontrivial classical solutions of (0.4), (0.2)–(0.3) of period P_m with $(\rho(0), u(0), P_m) = (0, 0, P_m)$. The period P_m depends on ν_0.

This result corresponds to that of Melrose and Pemberton [2], who, however, gave a different and less natural condition on g and did not prove regularity of the weak solution u.

Now let $\nu = 0$ and $\rho_0 \in g_u(0)^{-1} \Lambda$. Then there exists a family $(\rho(c), u(c), P_m)$ of nontrivial solutions of (0.4), (0.2)–(0.3) with $(\rho(0), u(0), P_m) = (\rho_0, 0, P_m)$. This result applies to the sine-Gordon equation

$$u_{tt} - u_{xx} = \rho \sin u \tag{0.6}$$

together with both boundary conditions (0.2). The dense set of ρ_0's in this case is exactly Λ. Moreover, the nontrivial solutions u are in $C^\infty(\mathbb{R} \times [0, \pi])$.

Let us return to our main result concerning equation (0.1). If $\lambda_0 = n^2$, $n \in \mathbb{N}$ for $(0.2)_{\mathrm{D}}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for $(0.2)_{\mathrm{N}}$, our bifurcation result is well known, since the problem can be reduced to an ordinary semilinear Sturm–Liouville problem. The branches emanating at $(n^2, 0, P)$ consist of solutions u which do not depend on t. Thus P is arbitrary. Due to a result of P. Rabinowitz [5] all these branches exist globally. Therefore we consider only the cases when $\lambda_0 \neq n^2$. If we restrict ourselves to the boundary conditions $(0.2)_N$, and if f does not depend on x, we can reduce (0.1) to an ordinary differential equation by seeking solutions which are independent of x. The ordinary differential equation $\ddot{u} - \lambda u = f(\lambda, u)$ has for every $\lambda < 0$ nontrivial periodic solutions of arbitrarily small amplitude, the periods converging to $2\pi/(-\lambda)^{1/2}$ if the amplitudes tend to zero. Thus in this case every point $(\lambda, 0, 2\pi/(-\lambda)^{1/2})$ with $\lambda < 0$ is a bifurcation point of (0.1), $(0.2)_N$, (0.3). If f depends on x, however, this argument fails.

Suppose $\lambda_0 \neq n^2$ and $f \equiv 0$. Then the linear Dirichlet-problem (0.1)–(0.3) has nontrivial solutions $(\lambda_0, c \sin nx \cos(n^2 - \lambda_0)^{1/2}\, t, 2\pi/(n^2 - \lambda_0)^{1/2})$ (or $(\lambda_0, c \cos nx \cos(n^2 - \lambda_0)^{1/2})\, t, 2\pi/(n^2 - \lambda_0)^{1/2})$ for the Neumann case) for each $n \in \mathbb{N}$ ($n \in \mathbb{N}_0$) with $\lambda_0 < n^2$. The question which naturally arises is whether at least one of these infinitely many branches emanating at $(\lambda_0, 0, 2\pi/(n^2 - \lambda_0)^{1/2})$ persists in a perturbed form for $f \not\equiv 0$. The affirmative answer to this question is only given for a dense set Λ of λ_0's. The difficulties which arise on the complementary set $\mathbb{R}\backslash\{n^2\}\backslash\Lambda$ will be explained later.

We restrict ourselves to nonlinearities $f(\lambda, x, u)$ depending only on the unknown function u and not on u_x or u_t. Aside from the technical difficulties which arise due to the dependence on the derivatives of u (see the end of Section 1) there is a striking change in the results for (0.1) if the nonlinearity depends also on u_t. For example, consider the equation

$$u_{tt} - u_{xx} - \lambda u = u^2 u_t\,. \tag{0.7}$$

For $\lambda = n^2$ the problem (0.7), (0.2)–(0.3) has the time independent solutions $(n^2, c \sin nx, P)$ or $(n^2, c \cos nx, P)$, while for $\lambda \neq n^2$ it has no solution but the trivial one. So the nonlinearity $u^2 u_t$ does not allow any nontrivial branch of the linearized problem to persist for $\lambda \neq n^2$.

In Section 1 we consider the linearized problem ($f \equiv 0$). We show the existence of a dense set Λ in $\mathbb{R}$ such that to any $\lambda_0 \in \Lambda$ there exists a minimal period P_m so that the linear problem with $\lambda = \lambda_0$ has only two linearly independent solutions of that period P_m. By a decomposition of our basic function space into the kernel of the linearized problem and the orthogonal complement we get an equivalent system of two equations to our semilinear problem (0.1)–(0.3), the period being fixed at P_m. The equation in the kernel is usually called the bifurcation equation. This well known method, which is due to Lyapunov and Schmidt, will be carried out in Section 2.

The difficulties which arise on the complementary set $\mathbb{R}\backslash\{n^2\}\backslash\Lambda$ are the following: either the bifurcation equation has to be solved in an infinite dimensional kernel or the equation in the orthogonal complement gives rise to a small divisor problem. Both cases seem to be rather difficult to treat and it is an open question whether the complementary set $\mathbb{R}\backslash\{n^2\}\backslash\Lambda$ contains any λ_0 such that $(\lambda_0, 0, P)$ is a bifurcation point of solutions of (0.1)–(0.3) for some period P.

In our approach we fix the period P_m and consider only λ as a bifurcation parameter. If the period P is varied, as it is for instance in the case of Hopf-bifurcation of time-periodic solutions of semilinear parabolic problems, one gets involved in small divisor problems which seem to be extremely difficult since no uniform number theoretical estimates for the divisors exist for a continuum of periods.

Section 1

We fix $\lambda_0 \neq n^2$, $n \in \mathbb{N}_0$, and consider the linearized problems

$$L_{\lambda_0}u \equiv u_{tt} - u_{xx} - \lambda_0 u = 0, \tag{1.1}$$

$$u(t, 0) = u(t, \pi) = 0. \tag*{(1.2)$_\mathrm{D}$}$$

or

$$u_x(t, 0) = u_x(t, \pi) = 0, \tag*{(1.2)$_\mathrm{N}$}$$

$$u(t + P, x) = u(t, x). \tag{1.3}$$

By expanding in Fourier series we see that the only possible values for P to get nontrivial solutions are $2\pi n/(n_0^2 - \lambda_0)^{1/2}$, $n_0^2 > \lambda_0$. We fix $n_0 \in \mathbb{N}$ or $n_0 \in \mathbb{N}_0$ for $(1.2)_\mathrm{D}$ or $(1.2)_\mathrm{N}$ respectively and set $n = 1$, thus getting a fixed period $P = 2\pi/(n_0^2 - \lambda_0)^{1/2}$ in condition (1.3). We now specify the class of functions in which we admit solutions u. Let $Q = (0, P) \times (0, \pi)$ and $W_2^2(Q)$ the real Sobolev space over Q. We define

$$\mathscr{D}_D = \{u \in W_2^2(Q), u(t, 0) = u(t, \pi) = 0, u(0, x) = u(P, x), u_t(0, x) = u_t(P, x)\}$$
$$\mathscr{D}_N = \{u \in W_2^2(Q), u_x(t, 0) = u_x(t, \pi) = 0, u(0, x) = u(P, x), u_t(0, x) = u_t(P, x)\}.$$

These definitions make sense since u is continuous on $[0, P] \times [0, \pi]$ and the functions u_t, u_x are continuous in t for fixed x and continuous in x for fixed t on $[0, P]$ and $[0, \pi]$ respectively. Generalizing the notion of a solution we replace (1.1)–(1.3) by

$$L_{\lambda_0}u = 0, \qquad u \in \mathscr{D}_D, \tag*{(1.4)$_\mathrm{D}$}$$

$$L_{\lambda_0}u = 0, \qquad u \in \mathscr{D}_N. \tag*{(1.4)$_\mathrm{N}$}$$

The linear differential operator L_{λ_0} acts in $L_2(Q)$ and is endowed with the domains of definitions $\mathscr{D}_D$ and $\mathscr{D}_N$. (Rigourously we should distinguish between L_{λ_0} with domain $\mathscr{D}_D$ and L_{λ_0} with domain $\mathscr{D}_N$, but we don't for the sake of simplicity of presentation. Also in what follows we write $\mathscr{D}$ to denote either $\mathscr{D}_D$ or $\mathscr{D}_N$

when it is not necessary to distinguish between the two cases.) The two domains can also be characterized in the following way by using Fourier series:

$$\mathscr{D}_D = \Big\{ u(t, x) = \mathrm{Re} \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} c_{kn} \sin kx \exp[in(n_0^2 - \lambda_0)^{1/2} t],\ c_{kn} \in \mathbb{C},$$

$$\| u \|_{\mathscr{D}}^2 = \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} | c_{kn} |^2 (k^2 + n^2)^2 < \infty \Big\} \qquad (\mathrm{Re} = \text{real part}),$$

$$\mathscr{D}_N = \Big\{ u(t, x) = \mathrm{Re} \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} c_{kn} \cos kx \exp[in(n_0^2 - \lambda_0)^{1/2} t],\ c_{kn} \in \mathbb{C},$$

$$\| u \|_{\mathscr{D}}^2 = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} | c_{kn} |^2 (k^2 + n^2)^2 < \infty \Big\}, \tag{1.5}$$

the usual Sobolev norm in $W_2^2(Q)$ being equivalent to $\| \ \|_{\mathscr{D}}$ in both cases. obviously (1.4) is satisfied for a nontrial u if and only if

$$k^2 - n^2(n_0^2 - \lambda_0) - \lambda_0 = 0 \tag{1.6}$$

for some $(k, n) \in \mathbb{N} \times \mathbb{Z}$ or $(k, n) \in \mathbb{N}_0 \times \mathbb{Z}$ for $(1.4)_{\mathrm{D}}$ or $(1.4)_{\mathrm{N}}$ respectively. In the following we distinguish two cases:

1. λ_0 is irrational. Then $(k, n) = (n_0, \pm 1)$ are the only solutions of (1.6). If $\ker(L_{\lambda_0})$ denotes the kernel of the operator L_{λ_0} acting in $L_2(Q)$ with domain $\mathscr{D}$, we have $\dim \ker(L_{\lambda_0}) = 2$ in this case.

2. λ_0 is rational, i.e. $\lambda_0 = p/q$ with $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$, p and q being relatively prime. Then (1.6) is equivalent to

$$q^2k^2 - (q^2n_0^2 - pq)\, n^2 = pq. \tag{1.7}$$

(Observe that we assumed $q^2n_0^2 - pq > 0$.) The solution set of this diophantine equation (1.7) is finite or infinite depending on whether $q^2n_0^2 - pq$ is the square of an integer or not. Let $q^2n_0^2 - pq = r^2$ for some $r \in \mathbb{N}$. Then (1.7) can be written as

$$(qk + rn)\,(qk - rn) = pq,$$

which obviously has only finitely many solutions $(k, n) \in \mathbb{N} \times \mathbb{Z}$ or $\mathbb{N}_0 \times \mathbb{Z}$. Next let $q^2n_0^2 - pq \neq r^2$, $r \in \mathbb{N}_0$. To treat this we first solve Fermat's equation

$$s^2 - (q^2n_0^2 - pq)\, t^2 = 1, \qquad (s, t) \in \mathbb{N} \times \mathbb{N},$$

which is possible since $q^2n_0^2 - pq \neq 0$ is not a square of an integer (see e.g. [1]). Now it is easily verified that the infinite sequence in $\mathbb{N}_0 \times \mathbb{N}$ defined by $(k_1, n_1) = (n_0, 1)$,

$$k_{j+1} = sk_j + (qn_0^2 - p)\,tn_j$$
$$n_{j+1} = qtk_j + sn_j$$

is in the solution set of (1.7).

We summarize:

PROPOSITION 1.1. *If* $q^2n_0^2 - pq = r^2$ *for some* $r \in \mathbb{N}$, *then* $\dim\ker(L_{\lambda_0}) < \infty$; *if* $q^2n_0^2 - pq \neq r^2$, $r \in \mathbb{N}_0$, *then* $\dim\ker(L_{\lambda_0}) = \infty$.

In the following we only consider the case when $\dim\ker(L_{\lambda_0}) < \infty$. By a suitable choice of n_0, i.e. of the prescribed period P, we can reduce the dimension of the kernel to two. For this purpose we define

$$\Lambda_D = \left\{\frac{p}{q} \in \mathbb{Q}, \frac{p}{q} \neq n^2, n \in \mathbb{N}_0, q^2n_0^2 - pq = r^2 \text{ for some } (n_0, r) \in \mathbb{N} \times \mathbb{N}\right\},$$

$$\Lambda_N = \left\{\frac{p}{q} \in \mathbb{Q}, \frac{p}{q} \neq n^2, n \in \mathbb{N}_0, q^2n_0^2 - pq = r^2 \text{ for some } (n_0, r) \in \mathbb{N}_0 \times \mathbb{N}\right\}.$$

Let $\lambda_0 \in \Lambda$ (where $\Lambda = \Lambda_D$ or Λ_N), p and q being relatively prime. Obviously there are only finitely many pairs $(n_0, r) \in \mathbb{N} \times \mathbb{N}$ or $\mathbb{N}_0 \times \mathbb{N}$ which satisfy the equation $q^2n_0^2 - pq = r^2$. The corresponding periods P are given by $2\pi/(n_0^2 - \lambda_0)^{1/2} = 2\pi q/r$. If we choose n_0 such that the corresponding period P_m is minimal among these finitely many periods, then all solutions of (1.7) are given by $(k, n) = (n_0, \pm 1)$. Indeed, assume there is a solution $(k, n) \neq (n_0, \pm 1)$. First of all $k = n_0$ if and only if $|n| = 1$. Our assumption on λ_0 excludes $n = 0$. Thus $|n| > 1$, which implies $q^2k^2 - pq = r^2n^2 > r^2$. Choosing $\tilde{n}_0 = k$, we get a period $\tilde{P} < P_m$, contradicting our assumption on P_m.

PROPOSITION 1.2. *Let* $\lambda_0 \in \Lambda$. *Then there exists a minimal period* P_m *such that the kernel of* L_{λ_0} *is two-dimensional. It is spanned by* $\sin n_0x \sin(2\pi/P_m)\,t$, $\sin n_0x \cos(2\pi/P_m)\,t$ *for* $(1.4)_D$ *or* $\cos n_0x \sin(2\pi/P_m)\,t$, $\cos n_0x \cos(2\pi/P_m)\,t$ *for* $(1.4)_N$ $(P_m = 2\pi/(n_0^2 - \lambda_0)^{1/2})$.

We give explicit subsets of Λ_D and Λ_N which are dense in $\mathbb{R}$:

$$\Lambda_D' = \left\{\frac{2sn_0 - 1}{s^2}, s \in \mathbb{Z}\backslash\{0\}, n_0 \in \mathbb{N}, sn_0 \neq 1\right\} \subset \Lambda_D,$$

$$\Lambda_N' = \left\{\frac{2sn_0 - 1}{s^2}, s \in \mathbb{Z}\backslash\{0\}, n_0 \in \mathbb{N}_0, sn_0 \neq 1\right\} \subset \Lambda_N.$$

We finish Section 1 by considering the linear inhomogeneous problems

$$L_{\lambda_0} u = g, \qquad u \in \mathscr{D}_D, \tag{1.8$_\mathrm{D}$}$$

$$L_{\lambda_0} u = h, \qquad u \in \mathscr{D}_N, \tag{1.8$_\mathrm{N}$}$$

for $\lambda_0 \in \Lambda_D$ and $\lambda_0 \in \Lambda_N$, $P = P_m$ respectively. Obviously g and h have to be an element of the orthogonal complement $\ker(L_{\lambda_0})^\perp$ in $L_2(Q)$. If $P_m = 2\pi/(n_0^2 - \lambda_0)^{1/2}$,

$$g(t, x) = \mathrm{Re} \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} d_{kn} \sin kx \exp[in(n_0^2 - \lambda_0)^{1/2} t]$$

and

$$h(t, x) = \mathrm{Re} \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} d_{kn} \cos kx \exp[in(n_0^2 - \lambda_0)^{1/2} t], \qquad \sum_k \sum_n | d_{kn} |^2 < \infty,$$

then necessarily $d_{n_0,\pm 1} = 0$. Expanding u in the same way as g and h respectively (see (1.5)) the coefficients c_{kn} of the solution u are given by

$$c_{kn} = \frac{d_{kn}}{k^2 - n^2(n_0^2 - \lambda_0) - \lambda_0} = \frac{q^2}{q^2k^2 - r^2n^2 - pq} d_{kn} \tag{1.9}$$

for $(k, n) \neq (n_0, \pm 1)$. This relation implies

$$| c_{kn} | \leqslant q \, | d_{kn} |, \qquad (k, n) \neq (n_0, \pm 1). \tag{1.10}$$

This estimate can't be improved in the sense that it regularizes the solution u. Consider for example the infinite dimensional subspace of $L_2(Q)$ defined by

$$V_D = \left\{ u(t, x) = \sum_k \sum_n c_{kn} \sin kx \exp[in(n_0^2 - \lambda_0)^{1/2} t], qk = r \, | n | \right\}.$$

Then

$$c_{kn} = -\frac{q}{p} d_{kn} \qquad \text{for } qk = r \, | n | . \tag{1.11}$$

This means that the "formal" solution u of $(1.8)_D$

$$u(t, x) = \mathrm{Re} \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} c_{kn} \sin kx \exp[in(n_0^2 - \lambda_0)^{1/2} t]$$

with coefficients c_{kn} given by (1.9) (which is in fact a "weak" solution) is only in $L_2(Q)$ if g is only in $L_2(Q)$. We gain no regularity by solving $(1.8)_\mathrm{D}$. The same argument holds also for $(1.8)_\mathrm{N}$. Since we are looking for solutions in the domains $\mathscr{D}_D$ and $\mathscr{D}_N$, we assume that $g \in \mathscr{D}_D \cap \ker(L_{\lambda_0})^\perp$ and $h \in \mathscr{D}_N \cap \ker(L_{\lambda_0})^\perp$. The estimate (1.10) then yields:

PROPOSITION 1.3. *Let $\lambda_0 \in \Lambda$ and $P = P_m$. For any $g \in \mathscr{D}_D \cap \ker(L_{\lambda_0})^\perp$ and $h \in \mathscr{D}_N \cap \ker(L_{\lambda_0})^\perp$ we have exactly one solution $u \in \mathscr{D}_D \cap \ker(L_{\lambda_0})^\perp$ and $u \in \mathscr{D}_N \cap \ker(L_{\lambda_0})^\perp$ of* $(1.8)_{\mathrm{D}}$ *and* $(1.8)_{\mathrm{N}}$ *respectively. Furthermore*

$$\| u \|_{\mathscr{D}} \leqslant q \| g \|_{\mathscr{D}}, \qquad \| u \|_{\mathscr{D}} \leqslant q \| h \|_{\mathscr{D}} \tag{1.12}$$

where q is the denominator of $\lambda_0 = p/q \in \Lambda$.

These sharp estimates (1.12) show that we gain no derivatives in inverting L_{λ_0} on $\ker(L_{\lambda_0})^\perp$. Therefore we are unable to treat (0.1) for nonlinearities which also depend on the derivatives of u.

SECTION 2

We write down the nonlinear problems as follows

$$L_{\lambda_0}u - \mu u = F(\lambda, u), \qquad u \in \mathscr{D}_D, \quad \mu = \lambda - \lambda_0, \tag*{$(2.1)_{\mathrm{D}}$}$$

$$L_{\lambda_0}u - \mu u = F(\lambda, u), \qquad u \in \mathbb{N}_N. \tag*{$(2.1)_{\mathrm{N}}$}$$

The nonlinear operator F is naturally given by

$$F(\lambda, u)(t, x) = f(\lambda, x, u(t, x)). \tag{2.2}$$

We assume:

$$f \in C^3(\mathbb{R} \times [0, \pi] \times \mathbb{R}), \qquad f(\lambda, x, 0) = f_u(\lambda, x, 0) = 0, \tag{2.3}$$

$$f_x(\lambda, 0, u) = f_x(\lambda, \pi, u) = 0, \tag*{$(2.3)_{\mathrm{N}}$}$$

the latter only in case of boundary conditions $(0.2)_{\mathrm{N}}$. It is well known that by the assumptions (2.3) f indices via (2.2) an everywhere defined operator $F\colon \mathbb{R} \times W_2^2(Q) \to W_2^2(Q)$ which is continuously Frechet-differentiable with respect to (λ, u) and satisfies

$$F(\lambda, 0) = 0, \qquad D_{(\lambda,u)}F(\lambda, 0) = 0, \tag{2.4}$$

(see e.g. [3], Chap. I, Section 2). Since periodicity and the boundary conditions are preserved by $F(\lambda, \cdot)$, its restriction to $\mathscr{D}_D$ and $\mathscr{D}_N$ induce operators in $\mathscr{D}_D$ and $\mathscr{D}_N$ respectively with the same properties.

PROPOSITION 2.1. *$F\colon \mathbb{R} \times \mathscr{D} \to \mathscr{D}$ is an everywhere defined continuously Frechet-differentiable operator satisfying* (2.4).

Let $\lambda_0 \in \Lambda$ and $P = P_m$ such that $\dim \ker(L_{\lambda_0}) = 2$. We define the orthogonal projector

$$P_0\colon \mathscr{D} \to \ker(L_{\lambda_0}) \subset \mathscr{D} \tag{2.5}$$

as follows: $P_0 u = (u, \varphi^+)\varphi^+ + (u, \varphi^-)\varphi^-$, where (,) denotes the scalar product in the complex space $L_2(Q)$ and

$$\varphi^\pm = \left(\frac{2}{P}\right)^{1/2} \sin n_0 x \exp[\pm i(n_0{}^2 - \lambda_0)^{1/2}\, t], \tag{2.6$_\text{D}$}$$

$$\begin{aligned} \varphi^\pm &= \left(\frac{2}{P}\right)^{1/2} \cos n_0 x \exp[\pm i(n_0{}^2 - \lambda_0)^{1/2}\, t], && \text{if} \quad n_0 \in \mathbb{N}, \\ \varphi^\pm &= \frac{1}{(\pi P)^{1/2}} \exp[\pm i(-\lambda_0)^{1/2}\, t], && \text{if} \quad n_0 = 0. \end{aligned} \tag{2.6$_\text{N}$}$$

Decomposing any $u \in \mathscr{D}$ by

$$u = P_0 u + Q_0 u = v + w, \qquad Q_0 = I - P_0\,, \tag{2.7}$$

(I denoting the identity), we get a system being equivalent to (2.1):

$$\begin{aligned} &\text{(a)} & L_{\lambda_0} w - \mu w &= Q_0 F(\lambda, v + w) \\ &\text{(b)} & -\mu v &= P_0 F(\lambda, v + w). \end{aligned} \tag{2.8}$$

This decomposition is called the Lyapunov–Schmidt decomposition for equation (2.1). If we write (2.8a) as

$$w - \mu L_{\lambda_0}^{-1} w - L_{\lambda_0}^{-1} Q_0 F(\lambda, v + w) = 0, \qquad w \in Q_0 \mathscr{D}, \tag{2.9}$$

Propositions 1.3, 2.1, and the implicit function theorem yield:

PROPOSITION 2.2. *There exist $\delta_1, \delta_2 < 0$ such that equation* (2.8a) *has a unique solution $w = w(\mu, v) \in \mathscr{D}$ whenever $|\mu| < \delta_1$, $\|v\| < \delta_2$. ($\|\ \|$ is some fixed norm in* $\ker(L_{\lambda_0})$). *Moreover, $w(\mu, 0) = 0$, w is continuously Frechet-differentiable in $\mathscr{D}$ with respect to (μ, v) and $D_{(\mu, v)} w(\mu, 0) = 0$.*

We substitute this solution $w = w(\mu, v)$ into equation (2.8b) and get the bifurcation equation:

$$\mu v + P_0 F(\lambda, v + w(\mu, v)) = 0, \qquad |\mu| < \delta_1\,, \qquad \|v\| < \delta_2\,. \tag{2.10}$$

Clearly, $v = 0$ is a solution for all $|\mu| < \delta_1$. We are only interested in nontrivial solutions (μ, v) bifurcating at (0, 0).

Let $v = c_+\varphi^+ + c_-\varphi^-$. We have to impose $c_- = \overline{c_+}$ so that v is real. By a phaseshift we can assume that $c_+ = c_- = c \in \mathbb{R}$, which implies the symmetry:

$$v\left(\frac{P_m}{2} - t, x\right) = v\left(\frac{P_m}{2} + t, x\right). \tag{2.11}$$

Since our basic equations are autonomous and contain only derivatives of u with respect to t of second order, the subspace of symmetric functions of $\mathscr{D}$ in the sense of (2.11) is invariant for the linear and nonlinear operators. By solving (2.8a) in the subspace of symmetric functions we get by uniqueness

$$w\left(\frac{P_m}{2} - t, x\right) = w\left(\frac{P_m}{2} + t, x\right) \tag{2.12}$$

for all solutions $w = w(\mu, v)$, whenever v satisfies (2.11). This implies that (2.10) is equivalent to the single equation in $\mathbb{R}$:

$$\mu c + (F(\lambda, v + w(\mu, v)), \varphi) = 0, \tag{2.13}$$

$\varphi = (\varphi^+ + \varphi^-)/2$, $v = c(\varphi^+ + \varphi^-)$, $c \in \mathbb{R}$, $|c| < \delta_3$ for some suitable $\delta_3 > 0$. Defining

$$\begin{aligned} N(\mu, c) &= \frac{1}{c}(F(\lambda, v + w(\mu, v)), \varphi) \qquad && \text{for } c \neq 0 \\ &= 0 && \text{for } c = 0, \end{aligned} \tag{2.14}$$

we can show

PROPOSITION 2.3. *The real valued function* $N: (-\delta_1, \delta_1) \times (-\delta_3, \delta_3) \to \mathbb{R}$ *as well as* $\partial N/\partial\mu = N_\mu(\mu, c)$ *are continuous and* $N(\mu, 0) = N_\mu(\mu, 0) = 0$.

In fact, the continuity of N follows easily by the properties of F and w (see Prop. 2.2). Now, for $c \neq 0$

$$\begin{aligned} \frac{\partial N}{\partial \mu} = \Big(&\frac{1}{c} f_\lambda(\lambda_0 + \mu, \cdot, v + w(\mu, v)) \\ &+ \frac{1}{c} f_u(\lambda_0 + \mu, \cdot, v + w(\mu, v))\, D_\mu w(\mu, v), \varphi\Big) \end{aligned}$$

is continuous, and for c tending to 0 we get the limit

$$\lim_{c \to 0} \frac{\partial N}{\partial \mu}(\mu, c) = (f_{\lambda u}(\lambda_0 + \mu, \cdot, 0) + f_{uu}(\lambda_0 + \mu, \cdot, 0)\, D_\mu w(\mu, 0), \varphi) = 0.$$

Thus, for $c \neq 0$, we reduced the bifurcation equation to the simple equation

$$\mu + N(\mu, c) = 0, \tag{2.15}$$

which has the solution $(\mu, c) = (0, 0)$. By $N_\mu(0, 0) = 0$ the implicit function theorem guarantees a local continuous family of solutions parameterized by c:

$$(\mu, c) = (\mu(c), c), \qquad |c| < \delta_4, \qquad \mu(0) = 0, \tag{2.16}$$

which are obviously nontrivial solutions of (2.13) emanating at (0, 0). We therefore proved

THEOREM 2.4. *Let $\lambda_0 \in \Lambda$. Then there locally exists a nontrivial continuous branch $(\lambda(c), u(c), P_m)$ in $\mathbb{R} \times \mathscr{D} \times \mathbb{R}_+$ of solutions of* (2.1) *which bifurcate from the trivial solutions at $(\lambda_0, 0, P_m)$.*

To finish the proof of our main result we have to show that under the assumption $f \in C^5(\mathbb{R} \times [0, \pi] \times \mathbb{R})$ the solutions $u(c)$ are not only in $\mathscr{D}$ but classical solutions of (0.1)–(0.3). The following procedure is similar to that in [6].

We assume in addition to (2.3), $(2.3)_N$

$$f \in C^{l+1}(\mathbb{R} \times [0, \pi] \times \mathbb{R}), \qquad l \geqslant 2, \tag{2.17}$$

and define

$$\mathscr{D}^l = \mathscr{D} \cap \left\{ u \in W_2{}^l(Q), \frac{\partial^j}{\partial t^j} u(0, x) = \frac{\partial^j}{\partial t^j} u(P, x), j = 0, ..., l-1 \right\}, \quad l \geqslant 2,$$

which, endowed with the Sobolev-norm of $W_2{}^l(\Omega)$, is a Hilbert-space. It is known that under the additional assumption (2.17) F restricted to $\mathbb{R} \times \mathscr{D} \cap W_2{}^l(Q)$ maps into $\mathscr{D} \cap W_2{}^l(Q)$ (see [3]), and it is easily verified that also

$$F: \mathbb{R} \times \mathscr{D}^l \to \mathscr{D}^l \tag{2.18}$$

holds. Furthermore F is continuously Frechet-differentiable on $\mathbb{R} \times \mathscr{D}^l$ and satisfies (2.4). Obviously $Q_0 F: \mathbb{R} \times \mathscr{D}^l \to \mathscr{D}^l \cap \ker(L_{\lambda_0})^\perp$ has the same properties as F.

PROPOSITION 2.5. *$L_{\lambda_0}^{-1}: \mathscr{D}^l \cap \ker(L_{\lambda_0})^\perp \to \mathscr{D}^l \cap \ker(L_{\lambda_0})^\perp$ is bounded with respect to the $W_2{}^l(Q)$-norm for all integers $l \geqslant 2$.*

The proof goes by induction. The case $l = 2$ is covered by Proposition 1.3. Let $L_{\lambda_0} u = u_{tt} - u_{xx} - \lambda_0 u = g \in \mathscr{D}^{l+1} \cap \ker(L_{\lambda_0})^\perp$ and assume that Proposition 2.5 holds for l. Then we know that $u \in \mathscr{D}^l \cap \ker(L_{\lambda_0})^\perp$. Let us define u for $t \in [kP, (k+1)P]$ by $u(t, x) = u(t - kP, x)$, $k \in \mathbb{Z}$. Thus u is defined on $\mathbb{R} \times [0, \pi]$ and the functions $u^h = (1/h)(u(t+h, x) - u(t, x))$ restricted to $[0, P] \times [0, \pi]$ are in $\mathscr{D}^l \cap \ker(L_{\lambda_0})^\perp$ for all $h \neq 0$. Furthermore they satisfy $L_{\lambda_0} u^h = g^h$, where g^h is defined in the same way. By assumption we have

$$\| u^h \|_{\mathscr{D}^l} \leqslant c_1 \| g^h \|_{\mathscr{D}^l} \leqslant c_2 \qquad \text{for all } h \neq 0, \tag{2.19}$$

the uniform bound c_2 being a consequence of $g \in \mathscr{D}^{l+1}$. If h tends to zero, estimates (2.19) imply that

$$u_t \in \mathscr{D}^l \qquad \text{and} \qquad \| u_t \|_{\mathscr{D}^l} \leqslant c_1 \| g_t \|_{\mathscr{D}^l} \tag{2.20}$$

holds. The identity $-u_{xx} = g + \lambda_0 - u_{tt}$ implies that $(\partial^{l+1}/\partial x^{l+1})\, u$ exists in the distributional sense, that it is a L_2-function, and that

$$\left\| \frac{\partial^{l+1}}{\partial x^{l+1}} u \right\|_{L_2(Q)} \leqslant c_3 \| g \|_{W_2^{l+1}(Q)}. \tag{2.21}$$

By (1.12), (2.20), and (2.21) we get $u \in \mathscr{D}^{l+1} \cap \ker(L_{\lambda_0})^\perp$ and

$$\| u \|_{W_2^{l+1}(Q)} \leqslant c_4 \| g \|_{W_2^{l+1}(Q)}. \tag{2.22}$$

Now, by the properties of

$$L_{\lambda_0}^{-1} Q_0 F: \mathbb{R} \times P_0 \mathscr{D}^l \times Q_0 \mathscr{D}^l \to Q_0 \mathscr{D}^l$$

application of the implicit function theorem to equation (2.9) yields

THEOREM 2.6. *Under the additional assumption* (2.17) *the bifurcating branch* $(\lambda(c), u(c), P_m)$ *guaranteed by Theorem* 2.4 *is continuous in* $\mathbb{R} \times \mathscr{D}^l \times \mathbb{R}_+$.

In order to get periodic solutions u defined on $\mathbb{R} \times [0, \pi]$ we define u for $t \in [kP, (k+1)P]$ by $u(t, x) = u(t - kP, x)$. By the definition of $\mathscr{D}^l$ and Sobolev's embedding theorem, $l = 4$ will suffice to guarantee classical solutions $u \in C^2(\mathbb{R} \times [0, \pi])$ of (0.1)–(0.3). If $f \in C^\infty(\mathbb{R} \times [0, \pi] \times \mathbb{R})$ then $u \in C^\infty(\mathbb{R} \times [0, \pi])$.

We finish with some remarks on the cases not considered here.

If λ_0 is irrational the kernel of L_{λ_0} is two-dimensional for any period $2\pi/(n_0^2 - \lambda_0)^{1/2}$. But there is a striking difference from the case when λ_0 is rational. Consider relation (1.9). By a well known number theoretical result for any $\epsilon > 0$ we can find $(k, n) \in \mathbb{N} \times \mathbb{N}$ such that $0 < | k^2 - n^2(n_0^2 - \lambda_0) - \lambda_0 | < \epsilon$. That implies that $L_{\lambda_0} \mid \ker(L_{\lambda_0})^\perp$ has no bounded inverse. In this sense equation (2.8a) gives rise to a "small divisor problem" which, in general, seems to be very difficult. First we need some information how fast with respect to (k, n) the "small divisor" tends to zero. There are number theoretical results in this direction, depending strongly on the irrational number λ_0. But in addition to this difficulty we don't know a generalized implicit function theorem which applies directly to our situation.

Let λ_0 be rational. If $\lambda_0 \notin \Lambda$, $\lambda_0 \neq n^2$, $n \in \mathbb{N}_0$, we know by Proposition 1.1 that the kernel of L_{λ_0}, if not trivial, is infinite-dimensional. (For the same reason we excluded the case $\lambda_0 = 0$, which, in case of boundary conditions $(0.2)_{\mathrm{D}}$, can't be reduced to an ordinary differential equation and is therefore an open problem.) Due to the lack of compactness it seems rather difficult to solve the infinite-dimensional bifurcation equation.

Note added in proof. If the nonlinearity depends also on the derivatives of u, i.e., $F(\lambda, u) = f(\lambda, x, u, u_x, u_t)$, $f_u = f_{u_x} = f_{u_t}(\lambda, x, 0, 0, 0) = 0$, then a sufficient condition to obtain a *formal* expansion of a nontrivial branch emanating at $(\lambda_0, 0, P_m)$ is the following: f is even in u_t or odd in the two variables (u, u_x). Indeed, the procedure as described in Section 2 yields that expansion of u and λ in terms of the parameter c which represents v in $\ker(L_{\lambda_0})$. (Choose the subspace of symmetric or antisymmetric functions of $\mathscr{D}$ respectively.) Similarly, the value of λ can be fixed at λ_0 and the period P can be expanded around P_m. (By a standard substitution $t \to t' = (P/P_m)t$ the period is normalized again.)

Since in both cases the bifurcation problem is reduced to a one-dimensional equation, these expansions in terms of the parameter c are unique. By the reasons explained at the end of Section 1 and at the end of the Introduction this method is purely formal. It is related to the results of Refs. [7, 8], where f must not depend on derivatives of u.

REFERENCES

1. H. DÖRRIE, "Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur," Ferdinand Hirt, Breslau, 1933.
2. R. B. MELROSE AND M. PEMBERTON, Periodic solutions of certain nonlinear autonomous wave equations, *Proc. Cambridge Philos. Soc.* **78** (1975), 137–143.
3. J. MOSER, A rapidly convergent iteration method and non-linear partial differential equations. I, *Ann. Scuola Norm. Sup. Pisa* **20** (1966), 265–315.
4. P. H. RABINOWITZ, Free vibrations for a semilinear wave equation, *Comm. Pure Appl. Math.*, to appear.
5. P. H. RABINOWITZ, "A Global Theorem for Nonlinear Eigenvalue Problems and Applications," Contr. Nonl. Funct. Anal., pp. 11–36, Academic Press, New York, 1971.
6. P. H. RABINOWITZ, Periodic solutions of nonlinear hyperbolic partial differential equations, *Comm. Pure Appl. Math.* **20** (1967), 145–205.
7. J. B. KELLER AND LU TING, Periodic vibrations of systems governed by nonlinear partial differential equations, *Comm. Pure Appl. Math.* **19** (1966), 371–420.
8. J. P. FINK, W. S. HALL, AND S. KHALILI, Perturbation expansions for some nonlinear wave equations, *SIAM J. Appl. Math.* **24** (1973), 575–595.